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Proximinality in Subspaces of c_0

G. Godefroy and V. Indumathi

Équipe d'Analyse, Boite 186, Université Paris VI,
4, Place Jussieu, Paris Cedex 05, 75252 France

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We say that a normed linear space X is a $R(1)$ space if the following holds: If Y is a closed subspace of finite codimension in X and every hyperplane containing Y is proximinal in X then Y is proximinal in X. In this paper we show that any closed subspace of c_0 is a $R(1)$ space.

Sometimes, but not always, finite codimensional subspaces Y of a Banach space X are proximinal when every hyperplane of X containing Y is itself proximinal (see [2]). When this happens, we say, following [5], that X is a $R(1)$ space. Non trivial examples of $R(1)$ spaces are usually obtained through some smoothness condition on the space X^* (see e.g. [5], Prop. 1). However, the space c_0 is easily seen to be a $R(1)$ space although its dual l^1 is very far from smooth. We show in this paper that the $R(1)$ property extends to subspaces of c_0. We will do so by using a remote form of smoothness in l^1, namely a strong form of subdifferentiabilty for norm attaining functionals in l^1.

We consider only real normed linear spaces. Let X be a normed linear space. Then X^* denotes its dual. The closed unit ball and the unit sphere of X are denoted by $B(X)$ and $S(X)$ respectively. By $NA(X)$, we denote the subset of X^*, consisting of all the norm attaining functionals on X. For $x \in X$, we set

$$M(x) = \{g \in S(X^*) : g(x) = \|x\|\}.$$

If Y is a subspace of X, we say that Y is proximinal in X if every $x \in X$ has a nearest element from Y. That is, there exists $z \in Y$ such that

$$\|x - z\| = d(x, Y) = \inf\{\|x - y\| : y \in Y\}.$$

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If Y is a closed subspace of X, the annihilator of Y, denoted by $Y^\perp$, is given by

$$Y^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in Y\}$$

Finally, if Q is a compact Hausdorff space, $C(Q)$ denotes the space of all real valued, continuous functions defined on Q with the sup norm.

In 1963, Garkavi [3] gave the following easily checked but useful characterization of proximinal subspaces of finite codimension in terms of the finite dimensional annihilator spaces.

THEOREM A (Garkavi [3]). *Let Y be a closed subspace of finite codimension in a normed linear space X. Then Y is proximinal if and only if for each $\Phi \in B((Y^\perp)^*)$, there exists $x \in X$ such that*

$$\|\Phi\| = \|x\|$$

and

$$\Phi(f) = f(x) \qquad \text{for all} \quad f \in Y^\perp.$$

It immediately follows from the above theorem that if Y is a proximinal subspace of finite codimension in X then $Y^\perp \subseteq NA(X)$ or equivalently every hyperplane containing Y is proximinal in X. However the converse is far from true as the following slight modification of an example of Phelps [6] shows. In fact, this example shows that any infinite dimensional $C(Q)$ space, where Q is a compact Haussdorff space, contains a subspace Y of codimension 2 such that $Y^\perp \subseteq NA(X)$ but Y is not proximinal in $C(Q)$. Before giving the example, we quote a characterization, due to Garkavi, of proximinal subspaces of finite codimension in $C(Q)$ that is needed to complete the example. In the following, $S(\mu)$ for $\mu \in (C(Q)^*)$ denotes the support of the measure μ.

THEOREM B (Garkavi [4]). *Let Y be a closed subspace of finite codimension in $C(Q)$. Then Y is proximinal if and only if the annihilator space $Y^\perp$ satisfies the following three conditions*:

(i) $S(\mu^+) \cap S(\mu^-) = \varnothing$ *for each* $\mu \in Y^\perp \backslash \{0\}$.

(ii) μ *is absolutely continuous with respect to* ν *on* $S(\nu)$ *for every pair* μ, ν *in* $Y^\perp \backslash \{0\}$.

(iii) $S(\nu) \backslash S(\mu)$ *is closed for each pair* μ, ν *in* $Y^\perp \backslash \{0\}$.

EXAMPLE [5]. Select a sequence (q_n) in Q with $q_n \neq q_m$ for $n \neq m$, which has a cluster point $q_0 \in Q$ with $q_0 \neq q_n$ for $n = 1, 2, \ldots$. Define $\mu, \nu \in (C(Q))^*$ by

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{q_n} + \delta_{q_0}$$

$$\nu = \sum_{n=1}^{\infty} \frac{1}{4^n} \delta_{q_n}$$

where δ_q denotes the evaluation functional at $q \in Q$. Let

$$Y = \{x \in C(Q) : \mu(x) = 0 \text{ and } \nu(x) = 0\}$$

Then $Y^{\perp}$ is the two dimensional subspace generated by μ and ν. If α is any scalar and $2^n > -\alpha$,

$$(\mu + \alpha\nu)(q_n) = \frac{1}{2^n} + \frac{\alpha}{4^n} > 0$$

This implies that for any $\lambda \in Y^{\perp}$, we have $S(\lambda^+) \cap S(\lambda^-) = \emptyset$ or equivalently $Y^{\perp} \subseteq NA(C(Q))$. However Y is not proximinal in $C(Q)$ since condition (ii) of Theorem B does not hold for μ and ν.

We recall that a finite dimensional real normed space is polyhedral if its unit ball has finitely many extreme points. We proceed with the following characterization of finite dimensional polyhedral spaces which is needed in the proof of our main Theorem 3, and which is a special case of ([8], Theorem 4.4.).

LEMMA 1. *Let E be a finite dimensional normed linear space. Then E is polyhedral if and only if for each $x \in S(E)$ there exists $\varepsilon(x) > 0$ such that $y \in S(E)$ and $\|x - y\| < \varepsilon(x)$ implies $M(y) \subseteq M(x)$.*

Proof. Assume E is polyhedral. Then $\operatorname{ext}(B(E))$ is a finite set $\{x_1, x_2, \ldots, x_k\}$. Select any $x \in S(E)$ and let $(y_n) \subseteq S(E)$ converge to x. It suffices to show that $M(y_n) \subseteq M(x)$ eventually.

Since $B(E)$ is the convex hull of its extreme points, we can write

$$y_n = \sum_{i=1}^{k} \mu_i^n x_i, \quad n = 1, 2 \ldots$$

where $\mu_i^n \geqslant 0$ for all $1 \leqslant i \leqslant k$ and $n \geqslant 1$, and moreover $\sum_{i=1}^{k} \mu_i^n = 1$ for all n.

Taking a subsequence if necessary, we can assume that $\lim_{n \to \infty} \mu_i^n = \mu_i$ exists for each i, $1 \leqslant i \leqslant k$. Since the sequence (y_n) converges to x we have

$$x = \sum_{i=1}^{k} \mu_i x_i$$

Let $\varepsilon(x) = \min\{\mu_i : \mu_i \neq 0\}$. Then $\varepsilon(x) > 0$ and there is a positive integer n_0 such that

$$\max_{1 \leqslant i \leqslant k} |\mu_i - \mu_i^n| < \varepsilon(x) \tag{1}$$

for all $n \geqslant n_0$. Now if $\Phi \in M(y_n)$ for some $n \geqslant n_0$, we have

$$\Phi(y_n) = \|y_n\| = 1 = \sum_{i=1}^{k} \mu_i^n \Phi(x_i)$$

and this together with (1) implies

$$\{i : \mu_i \neq 0\} \subseteq \{i : \mu_i^n \neq 0\} \subseteq \{i : \Phi(x_i) = 1\}.$$

Hence

$$\Phi(x) = \sum_{i=1}^{k} \mu_i \Phi(x_i) = \sum_{i=1}^{k} \mu_i = 1$$

and $\Phi \in M(x)$. That is, $M(y_n) \subseteq M(x)$ for all $n \geqslant n_0$.

Conversely assume that E satisfies the condition of the lemma. To show that E is polyhedral, we first observe that since E is finite dimensional, $S(E)$ is a compact set. This, together with our assumption about E, implies that we can get a finite subset F of $S(E)$ such that for any $y \in S(E)$, $M(y) \subseteq M(x)$ for some $x \in F$. Choose any $f \in S(E^*)$. Then by compactness of $B(E)$, $f \in M(y)$ for some $y \in S(E)$ and hence $f \in M(x)$ for some $x \in F$. Hence every $f \in S(E^*)$ attains its norm at some point of the finite set F of $S(E)$. This, in turn, implies that E^* is polyhedral. Hence E is polyhedral. This concludes the proof of Lemma 1. ▮

We now consider a closed subspace X of the sequence space c_0 and a closed subspace Y of X which is of finite codimension in X. We will now show that if $Y^\perp$ is contained in $NA(X)$, then $Y^\perp$ is polyhedral. In order to do this, we show that the finite dimensional space $Y^\perp$ satisfies the condition of Lemma 1.

LEMMA 2. *Let X be a closed subspace of c_0 and Y be a closed subspace of X that is of finite codimension in X. Assume further that the annihilator $Y^\perp$ of Y in X^* is contained in $NA(X)$. Then $Y^\perp$ is polyhedral.*

Proof. We will prove that if $g \in NA(X)$ with $\|g\| = 1$, then there exists $\varepsilon(g) > 0$ such that $h \in S(X^*)$ and $\|h - g\| < \varepsilon(g)$ would imply $M(h) \subseteq M(g)$. This together with Lemma 1 would complete the proof of this lemma.

Clearly, we only need to show that if $g \in NA(X)$ with $\|g\| = 1$ and $(h_n) \subseteq S(X^*)$ converges to g then $M(h_n) \subseteq M(g)$ eventually. We denote by Q the canonical quotient map from $l^1 = c_0^*$ onto X^*. The bidual of X is identified to the subspace $X^{\perp\perp}$ of $(l^1)^* = l_\infty$. Pick H_n in the unit sphere of l^1 such that $Q(H_n) = h_n$. Taking a subsequence if necessary, we may and do assume that the sequence H_n converges weak* to $G \in l_1$. We clearly have $Q(G) = g$.

For any H in l^1, we denote by

$$\mathrm{Supp}(H) = \{k \geqslant 1;\, H(e_k) \neq 0\}$$

where (e_k) is the natural basis of c_0. Using the notation $M(.)$ as defined above, we clearly have

$$M(h_n) = M(H_n) \cap X^{\perp\perp} \tag{2}$$

and

$$M(g) = M(G) \cap X^{\perp\perp} \tag{3}$$

If we denote by $t = (t_n)$ vectors in l_∞, we check easily that for any $H \in l_1$ we have

$$M(H) = \{t \in S(l_\infty);\, t_k = \mathrm{sign}(H(e_k)) \text{ for all } k \in \mathrm{Supp}(H)\} \tag{4}$$

We observe now that $G \in NA(c_0)$. In fact, since $g \in NA(X)$, G attains its norm at some point of the unit sphere of X. It follows easily that $\mathrm{Supp}(G)$ is a finite set. Hence there exists N such that for all $n \geqslant N$, we have $\mathrm{Supp}(G) \subset \mathrm{Supp}(H_n)$ and

$$\mathrm{sign}(H_n(e_k)) = \mathrm{sign}(G(e_k))$$

for all $k \in \mathrm{Supp}(G)$. Now it follows from (2) that $M(H_n) \subseteq M(G)$ for $n \geqslant N$ and then (3) and (4) show that the conditions of Lemma 1 are satisfied. This concludes the proof of Lemma 2. ∎

We can now present our main result.

THEOREM 3. *Every closed subspace of c_0 is a $R(1)$ space.*

Proof. Let X be a closed subspace of c_0 and Y be a closed subspace of finite codimension in X. Assume that the annihilator $Y^\perp$ of Y in X is contained in $NA(X)$. We need to show that Y is proximinal in X.

By Lemma 2, we know that the finite dimensional space $Y^{\perp}$ is polyhedral. Thus every extreme point of $B((Y^{\perp})^*)$, the unit ball of the dual space $(Y^{\perp})^*$, is in fact exposed. Select any such $\Phi \in \text{ext}(B(Y^{\perp})^*)$ and a linear functional $f \in S(Y^{\perp})$ that exposes Φ. Since $Y^{\perp} \subseteq NA(X)$, there is $x \in X$ such that $\|x\| = 1$ and $f(x) = \|f\| = 1$. Since f exposes Φ,

$$\Phi(h) = h(x) \qquad \text{for all} \quad h \in Y^{\perp}$$

and of course

$$\|\Phi\| = \|x\| = 1.$$

Thus Garkavi's criterion, given in Theorem A, is satisfied for each $\Phi \in \text{Ext } B((Y^{\perp})^*)$. Since $B((Y^{\perp})^*)$ is the convex hull of its extreme points, it easily follows that Garkavi's criterion holds for any Φ in the unit ball of $(Y^{\perp})^*$. Thus Y is proximinal in X by Theorem A, and this concludes the proof of Theorem 3. ∎

Remarks. (a) The set $NA(c_0)$ is equal to the space of finitely supported elements of l^1 and it is therefore a vector space of countable algebraic dimension. Hence if X is a subspace of c_0, $NA(X)$ is contained in a vector subspace of X^* of countable algebraic dimension. It follows from Baire category theorem that if Y is a subspace of X such that $Y^{\perp}$ is contained in $NA(X)$, then the codimension of Y in X is finite, and Y is proximinal in X by Theorem 3.

(b) It is easy to deduce from Theorem 3 that if X is a subspace of c_0, the following assertions are equivalent: (i) $NA(X)$ is a vector subspace of X^*. (ii) The intersection of two proximinal hyperplanes is proximinal. (iii) The intersection of a finite number of proximinal hyperplanes is proximinal. Any finite codimensional proximinal subspace X of c_0 satisfies these conditions. On the other hand, there is an hyperplane of c_0 which fails to satisfy them. Indeed, let

$$E = (1/2, 1/2, 1/4, 1/8, 1/16, \ldots)$$

be in l^1, and let $X = Ker(E)$. Moreover, let $g_1 = (1, 0, 0, 0, \ldots)$ and $g_2 = (0, 1, 0, 0, \ldots.)$ be the restrictions to X of the corresponding elements of l^1. It is easy to check that g_1 and g_2 belong to $NA(X)$, however $(g_1 + g_2) \notin NA(X)$. Very little seems to be known about Banach spaces which have an equivalent norm such that the set of norm attaining functionals for this norm is a vector subspace of the dual space. It follows from [1], [7] and James' characterization of reflexivity that no such norm exists on a non reflexive space with the Radon–Nikodym property.

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